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On the maximum thrust of a yacht by sailing close to wind

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SUMMARY
The maximum thrust produced by the combined action of sails and keel is considered. The theory is linearized and holds for sailing close to wind. A numerical example is given.

1. Introduction

In a former paper [1], the problem has been considered of the maximum thrust production of sails, while sailing close to the wind. By this is meant that the angle between the ship's course and the negative direction of the velocity of the prevailing wind is small of $O(\varepsilon)$ (ε is a small parameter). This implies that also the direction of the thrust makes an angle of $O(\varepsilon)$ with the negative wind direction. Under these circumstances we are able to optimize the thrust in a linearized theory because then the thrust and the induced resistance are of the same order of magnitude.

The keel of a sailing yacht is needed to neutralize, in combination with the hull, the sideforce and heeling moment produced by the sails. In general this is done by using the keel as a rigid lifting surface with a weight connected to it. However, we can also look at the keel as a lifting surface, which can be given an optimum shape in order to minimize its induced resistance, and thus optimize its action. Here we will consider the combined thrust production of sails and keel optimized in this way. It will be clear that then there is no fundamental difference between the keel and the sails, as far as the thrust production is concerned.

The essential condition for sailing is the existence of a difference in the velocities of air and water. In order that sails and keel can both be treated by a theory for "sailing close to wind", it is necessary that the angle between the velocities of air and water with respect to the yacht is small of $O(\varepsilon)$. To simplify the problem, we consider the keel and the submerged part of the hull as one lifting surface. In a linearized optimization theory, which will be used here, we may, without restricting generality, represent lifting surfaces by lifting lines. We stress that this theory is also valid for sails and keels of small aspect ratio; they only have to be lightly loaded, because then we may linearize.

The lifting line, representing the keel, will end at the boundary between air and water. In [1] it has been shown, that in the situation considered here we may represent the sails by *one* lifting line; this line will end at some distance from the water surface in order to simulate the gap between boom and hull. (Rather than simulating the gap between boom and hull, we have to simulate the one between foresail and deck, for, when both sails are represented by *one* lifting line, this is the smallest gap.) Thus the yacht is, as far as its thrust production is concerned, considered as two coupled lifting lines, one of which protrudes in the air, the other one in the water. The induced resistance is, by its very nature, taken into account in this theory, while determining the optimum circulation of the lifting lines.

In the state of uniform motion, the driving force acting on the yacht, and produced by sails and keel, is neutralized by the form drag, friction drag and wavemaking resistance, which together will be called the total resistance of the yacht. In this paper we will not try to find this uniform motion state, but only determine the maximum thrust under certain given circumstances. This means that we do not calculate the (maximum) speed of the yacht, so in the computations a reasonable choice has to be made for its magnitude.

Important data in this problem are the velocities of air and water with respect to the yacht, the span of sails and keel, the width of the gap between sails and deck, and the stability curve of the yacht. For some values of these parameters, we will determine the maximum thrust and the corresponding circulations of sails and keel.

It will turn out that in general the optimum circulation distributions, even in the upright position of the yacht, are *not* elliptical, as is sometimes believed. First this is a consequence of the reflecting property of the water surface. The second reason is that there are constraints on the actions of sails and keel, because their sideforces have to be equal and opposite, while their moments have to be balanced by the righting moment of the yacht.

Other authors (e.g. Tanner [2], Milgram [3], [4]) have already used the idea of replacing sails by lifting lines. Tanner used the lifting line representation to calculate the distribution of the loading, downwash velocity and induced drag of a "Finn" dinghy sail when sailing upright. Milgram used lifting line theory to minimize the induced resistance of sails. He observed that the effect of heeling on sail aerodynamics is negligible, as long as the heeling angle does not become too large. This was confirmed by the calculations in [1]. Milgram reduced the heeling moment, created by the sails, by an appropriate choice of the coefficients in the Fourier expansion of the circulation distribution ([4], Section 2). The idea of reducing the heeling moment and even the sideforce has been further worked out in [1]. There a consistent linearized optimization theory has been given, when constraints are imposed on the sideforce and heeling moment.

Curry already suggested ([5], p. 133) to use cambered centreboards in order to improve the dynamic action of the underwater ship. In an attractive introduction to the dynamics of sailing, Kay [6] explains that there is no essential difference between the underwater- and above-water-part of a sailing yacht when only the thrust production is considered.

The aim of this paper is to combine the mentioned ideas in a linearized optimization theory for both sails and keel in their indissoluble reciprocal relation.

2. Formulation of the problem

Consider a right-handed Cartesian coordinate system (XYZ) which is fixed to the yacht. The region $Z > 0$ is filled with air, the region $Z < 0$ with water.

In the YZ-plane we assume a line p which makes an angle β (the heeling angle) with the Z-axis. Along this line we have in the halfspace $Z \geqq 0$ a parameter $s \geqq 0$, which denotes

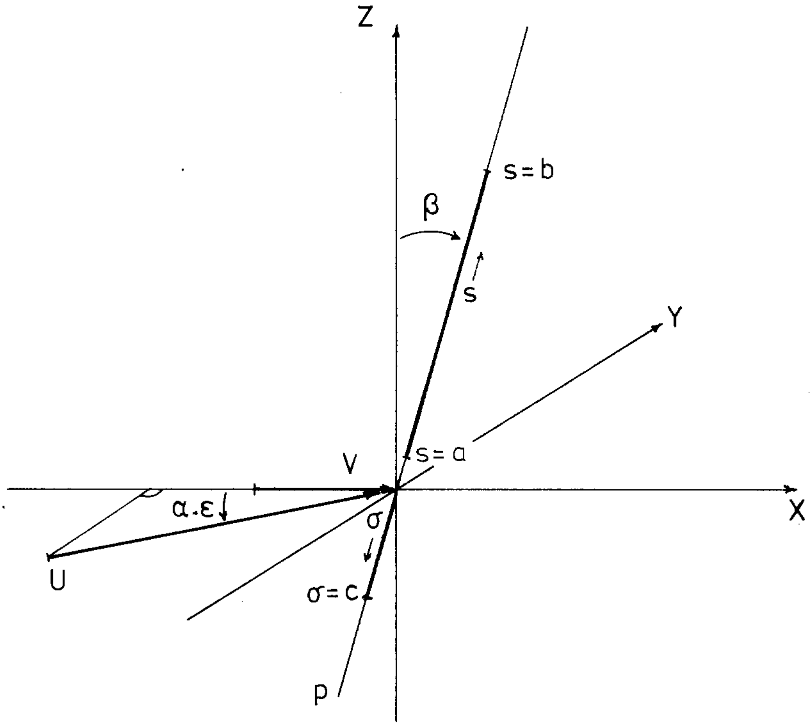

Figure 2.1. The lifting lines and the incoming flow.

the distance of a point of p to the origin, and in the halfspace $Z \leqq 0$ an analogous parameter $\sigma \geqq 0$.

The lifting line representing the sails is lying along p from $s = a$ to $s = b$, the line representing the keel extends from the origin to $\sigma = c$. The circulations $\Gamma^a(s)$ and $\Gamma^\omega(\sigma)$ of the lifting lines are taken positive if they are connected with a right-handed screw in the direction of increasing s or increasing σ respectively. Here we introduced the convention that quantities belonging to the air are given a superscript "a", those belonging to the water a superscript "ω".

The water has a uniform velocity of magnitude V, and is directed along the positive X-axis. The wind has a uniform velocity of magnitude U, and makes an angle $\varepsilon\alpha$ with the X-axis. Here ε is a small parameter, with respect to which the theory is linearized, while α is $O(\varepsilon^0)$.

The thrust T is the component of the total force acting on the system of lifting lines, which is directed along some prescribed line parallel to the (XY)-plane, making an angle $\varepsilon\alpha_1$ (α_1 of $O(\varepsilon^0)$) with the X-axis. It will be taken positive if T is directed into the halfspace $X < 0$. The sideforce F_s is the component of the total force perpendicular to T and parallel to the (XY)-plane, its positive direction being the positive direction of T, rotated clockwise (anticlockwise) over $\frac{1}{2}\pi$ radians for the sails (keel). The heeling moment M_h is the moment exerted around the working line of T, positive if it is connected as a right-handed screw with the positive direction of T.

Now, if we have chosen the velocities U and V, and an angle $\varepsilon\alpha$ between them, we do not yet know the exact direction of the total resistance R (we only know that it makes an angle of $O(\varepsilon)$ with the positive X-axis), and therefore we can not prescribe the direction ($\varepsilon\alpha_1$) in which we want to maximize the thrust. However, this difficulty can be overcome by the following reasoning. The thrust T is $O(\varepsilon^2)$, and it is neutralized by the total resistance which, therefore, is of the same order of magnitude. So if R and T are not exactly working in opposite direction, but are making an angle $\pi - O(\varepsilon)$, their resultant force is $O(\varepsilon^3)$, an order of magnitude which is not taken into account in our linearized theory. This makes it reasonable to formulate our problem as follows:

Find the direction, which makes an angle $\varepsilon\alpha_1$ with the negative X-axis, together with the heeling angle β, for which the thrust assumes its maximum value, while the following conditions are satisfied:

(*a*) up to and including $O(\varepsilon)$ there should be no resultant sideforce,

(*b*) if the yacht has a righting moment, depending on the angle of heel β, of strength $m^*(\beta)$, the heeling moment, exerted by sails and keel, should be in equilibrium with this righting moment.

In the following we will see that the thrust is up to and including quantities of $O(\varepsilon^2)$ independent of the direction in which it is demanded, as long as this direction makes an angle of $O(\varepsilon)$ with the X-axis. This is a consequence of the sideforces of sails and keel being equal and opposite. Hence in the above formulation the direction of the thrust can be chosen to be the negative X-direction.

We remark here that we have tacitly assumed that the yacht has enough longitudinal stability to neutralize the moment around an axis, normal to the thrust, exerted by driving forces of sails and keel, and which is $O(\varepsilon^2)$.

In this paper we will assume the water to have zero velocity in a coordinate system, which is at rest with respect to a fixed point on the shore (it is not "streaming"). This is no real restriction. The wind with respect to this coordinate system will be called the true wind $\bar{U}$. The angle between the ship's course and the direction of $\bar{U}$ is called the true angle of incidence $\bar{\alpha}$. In Fig. 2.2, the relation between $\bar{U}$, U (which we call the relative windspeed), $\bar{\alpha}$ and α (the relative angle of incidence) is shown.

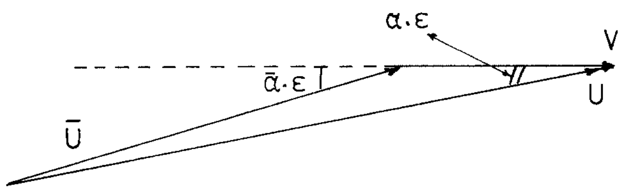

Figure 2.2. Relation between true and relative windspeed and angle of incidence.

From this figure we see that the relative angle of incidence is, in the case of sailing close to wind, always smaller than the true one, whereas the relative windspeed is always greater than the true one.

3. The optimization problem

We recapitulate some of the notations and results of [1], and will adapt these to the problem under consideration. A modification of the formulae of [1] will be necessary, because there it is not possible to determine the minimum induced resistance of a lifting line, when the sideforce is prescribed in the direction normal to the incoming flow (in fact this means that we prescribe the total aerodynamic force acting on the lifting line). Such a result will be needed here, so we shortly treat this optimization problem for the sails; the keel can be treated completely analogously. In this section we neglect the superscript "a".

In a right-handed Cartesian coordinate system XYZ, with respect to which the fluid is at rest, a lifting line moves with constant speed U in the negative X-direction. The shed free vorticity lies on a strip H, parallel to the X-axis, which makes an angle β with the Z-axis.

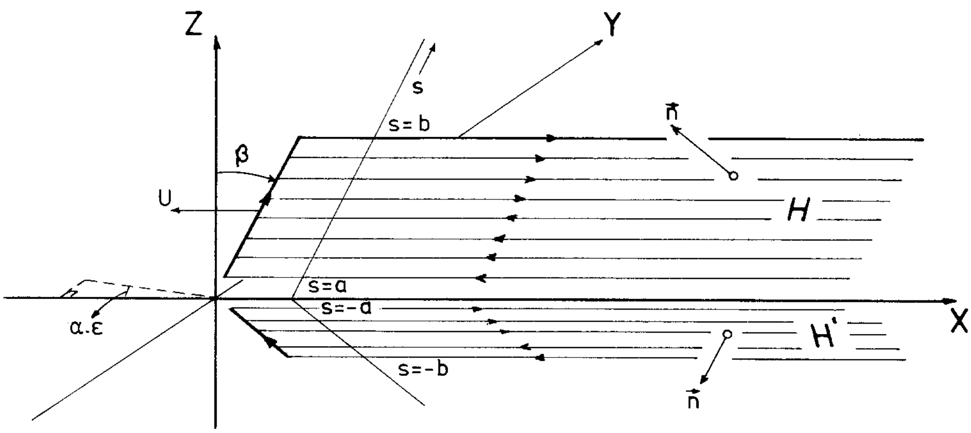

Figure 3.1. The lifting line and the strips H and H'.

At the water surface, supposed to be at the plane $Z = 0$, the boundary condition to be satisfied is that the normal velocity vanishes. In order to simulate this, the strip H is reflected in the plane $Z = 0$, its image being called H'. At the plane through H, which contains the X-axis, we have a parameter s which denotes the distance of a point of this plane to the X-axis. At the plane through H' the corresponding parameter has negative values. H extends from $s = a$ to $s = b$, so H' extends from $s = -a$ to $s = -b$ (Figure 3.1).

The circulation $\Gamma(s)$ of the lifting lines is taken positive if it is connected with a right-handed screw in positive s-direction, the free vorticity $\gamma(s)$ is positive if it is connected with a right-handed screw in the positive X-direction. Because of the symmetry we have $\Gamma(s) = \Gamma(-s)$, $\gamma(s) = -\gamma(-s)$. By virtue of Kelvin's law the total circulation of H, and hence of H', is zero.

The sideforce F_s, positive in positive Y-direction, and the heeling moment M_h, positive in negative X-direction, are prescribed:

$$F_s = \varepsilon \tilde{F}_s, \quad M_h = \varepsilon \tilde{M}_h, \tag{3.1}$$

where $\tilde{F}_s$ and $\tilde{M}_h$ are $O(\varepsilon^0)$.

Now, if the lifting line moves throught the fluid, a certain amount of work has to be performed per unit of time in order to overcome the induced resistance. We may also say that this work is used to build up the free vortex sheet, for no energy is radiated to infinity because of the incompressibility of the fluid. This vortex sheet adds per unit of time an amount of kinetic energy to the fluid equal to the performed amount of work. Posed in this way, we see that minimizing the induced resistance is the same as minimizing the kinetic energy put into the fluid per unit of time.

In the Trefftz region (far behind the lifting line), where the influence of the lifting line has disappeared, the fluid flow is two-dimensional and can be described by a potential function $\varphi(y, z)$. Here the amount of kinetic energy per unit of length is

$$E(\varphi) = \tfrac{1}{2}\rho \iint_{-\infty}^{\infty} \left\{ \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right\} dy\, dz, \tag{3.2}$$

where ρ is the density of the medium.

The constraints on the sideforce and the heeling moment can be written as

$$K_1(\varphi) \equiv +\rho U \cos\beta \int_{-b}^{b} [\varphi(s)]_+^- \, ds = -2\varepsilon \tilde{F}_s, \tag{3.3}$$

$$K_2(\varphi) \equiv +\rho U \int_{-b}^{b} |s| [\varphi(s)]_+^- \, ds = -2\varepsilon \tilde{M}_h, \tag{3.4}$$

where $[\varphi(s)]_+^-$ is the jump of φ over the strip H or H' and $[\varphi(s)]_+^- \equiv 0$ for $-a < s < a$.

We introduce the Lagrange multipliers λ_1 and $\lambda_2 l^{-1} \cos\beta$, where $l = b - a$; then we have to minimize the functional

$$G(\varphi) = E(\varphi) + \lambda_1 K_1(\varphi) + \lambda_2 l^{-1} \cos\beta\, K_2(\varphi). \tag{3.5}$$

If φ is the optimum potential function, the first variation G in φ has to vanish:

$$o = \delta G = -\rho \int_{-b}^{b} \left\{ \frac{\partial \varphi}{\partial n} - U \cos\beta \left(\lambda_1 + \lambda_2 \frac{|s|}{l} \right) \right\} [\delta\varphi(s)]_+^- \, ds, \tag{3.6}$$

where $\partial/\partial n$ means differentiation in the direction of the normal to H or H', which has been indicated in Fig. 3.1.

In [1] it has been shown that from (3.6) follows:

$$\frac{\partial \varphi}{\partial n} = U \cos\beta \left(\lambda_1 + \lambda_2 \frac{|s|}{l} \right), \qquad s \in (-b, -a) \cup (a, b). \tag{3.7}$$

A more general optimization theory is given in [7], of which this result is a special case.

We now introduce the functions ψ_1 and ψ_2, which satisfy

$$\left. \begin{aligned} & \frac{\partial \psi_1}{\partial n} = 1, \ \frac{\partial \psi_2}{\partial n} = |s|, \quad s \in \left(-\frac{a}{l} - 1, -\frac{a}{l} \right) \cup \left(\frac{a}{l}, \frac{a}{l} + 1 \right), \\ & \Delta \psi_i = 0, \ \psi_i \to 0 \ \text{for} \ y^2 + z^2 \to \infty. \end{aligned} \right\} \tag{3.8}$$

Then, by (3.7), we can write φ as

$$\varphi = Ul \cos \beta(\lambda_1 \psi_1 + \lambda_1 \psi_2). \tag{3.9}$$

The induced normal velocity $w_n(s)$ at the lifting line is half the induced normal velocity at the corresponding point of the strip in the Trefftz plane, so

$$w_n(s) = \frac{1}{2}\frac{\partial \varphi}{\partial n} = \tfrac{1}{2}U \cos \beta\left(\lambda_1 + \lambda_2 \frac{|s|}{l}\right), \qquad s \in (-b, -a) \cup (a, b). \tag{3.10}$$

The induced resistance R_i, positive in negative X-direction, becomes

$$\begin{aligned} R_i &= +\rho \int_a^b w_n(s)[\varphi(s)]_+^- \, ds \\ &= +\tfrac{1}{2}\rho U^2 l^2 \cos^2 \beta \int_{a/l}^{1+a/l} (\lambda_1 + \lambda_2 |s|)\{\lambda_1[\psi_1(s)]_+^- + \lambda_2[\psi_2(s)]_+^-\} \, ds. \end{aligned} \tag{3.11}$$

If we introduce the notation

$$I_{jk} = \int_{a/l}^{1+a/l} [\psi_j(s)]_+^- s^k \, ds, \qquad j = 1, 2, \ k = 0, 1, \tag{3.12}$$

then R_i can be written as

$$R_i = \tfrac{1}{2}\rho U^2 l^2 \cos^2 \beta\{\lambda_1^2 I_{10} + \lambda_1 \lambda_2 (I_{20} + I_{11}) + \lambda_2^2 I_{21}\}. \tag{3.13}$$

In order to obtain simple formulae, we write the prescribed $\tilde{F}_s$ and $\tilde{M}_h$ as

$$\tilde{F}_s = -\mu_1 \rho U^2 l^2 \cos^2 \beta I_{10}, \quad \tilde{M}_h = -\mu_2 \rho U^2 l^3 \cos \beta I_{11}, \tag{3.14}$$

by which the factors μ_1 and μ_2 are defined, and can be computed.

By (3.3) and (3.4) λ_1 and λ_2 can be expressed in μ_1 and μ_2, and hence are known. Then the induced resistance R_i becomes

$$R_i = -\tfrac{1}{2}\varepsilon^2 \rho U^2 l^2 D^{-1} \cos^2 \beta\{\mu_1^2 I_{10}^2 I_{21} - \mu_1 \mu_2 I_{10} I_{11}(I_{11} + I_{20}) + \mu_2^2 I_{10} I_{11}^2\}, \tag{3.15}$$

where $D = I_{11} I_{20} - I_{10} I_{21}$. The corresponding spanwise circulation distribution $\Gamma(s)$ is

$$\begin{aligned} \Gamma(s) &= [\varphi(s)]_+^- \\ &= UlD^{-1} \cos \beta\{(\mu_2 I_{11} I_{20} - \mu_1 I_{10} I_{21})[\psi_1(s)]_+^- + I_{10} I_{11}(\mu_1 - \mu_2)[\psi_2(s)]_+^-\}. \end{aligned} \tag{3.16}$$

Now we can easily determine the maximum thrust, under the mentioned constraints, in a direction which makes an angle $\varepsilon\alpha_1$ (α_1 of $O(\varepsilon^0)$) with the X-axis, of a lifting line moving along H. We saw that the circulation $\Gamma(s)$ in (3.16) leaves behind minimum kinetic energy, so this is the circulation we need. The force component in the desired direction is simply the projection of the sideforce F_s in (3.14) and the induced resistance R_i in (3.15) on this direction:

$$T = R_i + \alpha_1 F_s$$
$$= -\tfrac{1}{2}\varepsilon^2 \rho U^2 l^2 D^{-1} \cos^2\beta\{\mu_1^2 I_{10}^2 I_{21} - \mu_1\mu_2 I_{10} I_{11}(I_{11} + I_{20})$$
$$+ \mu_2^2 I_{10} I_{11}^2 + 2\mu_1\alpha_1 D I_{10}\}. \tag{3.17}$$

With respect to the keel one remark has to be made. Because the line representing the keel ends at the water surface, Kelvin's law does not apply to the strips H and H' separately. Of course Kelvin's law is satisfied for H and H' together, because of $\gamma(s) = = -\gamma(-s)$.

4. The two coupled lifting lines

We defined the still unknown direction in which the thrust T has to be optimized to make an angle $\varepsilon\alpha_1$ (α_1 of $O(\varepsilon^0)$) with V. Then the angle between T and U is $(\alpha - \alpha_1)\varepsilon$.

From (3.17) we now get the maximum thrust, delivered by the sails and the keel:

$$T = -\tfrac{1}{2}\varepsilon^2 \cos^2\beta\left[\frac{\rho^a U^2(b-a)^2}{D^a}\{\mu_1^2(I_{10}^a)^2 I_{21}^a - \mu_1\mu_2 I_{10}^a I_{11}^a(I_{11}^a + I_{20}^a)\right.$$
$$+ \mu_2^2 I_{10}^a(I_{11}^a)^2 + \mu_1(\alpha - \alpha_1)2D^a I_{10}^a\} + \frac{\rho^\omega V^2 c^2}{D^\omega}\{\nu_1^2(I_{10}^\omega)^2 I_{21}^\omega$$
$$\left. - \nu_1\nu_2 I_{10}^\omega I_{11}^\omega(I_{11}^\omega + I_{20}^\omega) + \nu_2^2 I_{10}^\omega(I_{11}^\omega)^2 + \nu_1\alpha_1 2D^\omega I_{10}^\omega\}\right]. \tag{4.1}$$

Here ν_1 and ν_2 play the same role for the keel, as μ_1 and μ_2 do for the sails, see (3.14).

The equilibrium conditions for the sideforces and moments yield the following equations:

$$\mu_1\rho^a U^2(b-a)^2 I_{10}^a - \nu_1\rho^\omega V^2 c^2 I_{10}^\omega = 0, \tag{4.2}$$
$$\mu_2\rho^a U^2(b-a)^3 \cos\beta I_{11}^a + \nu_2\rho^\omega V^2 c^3 \cos\beta I_{11}^\omega + \rho^\omega V^2 c^3 m(\beta) = 0, \tag{4.3}$$

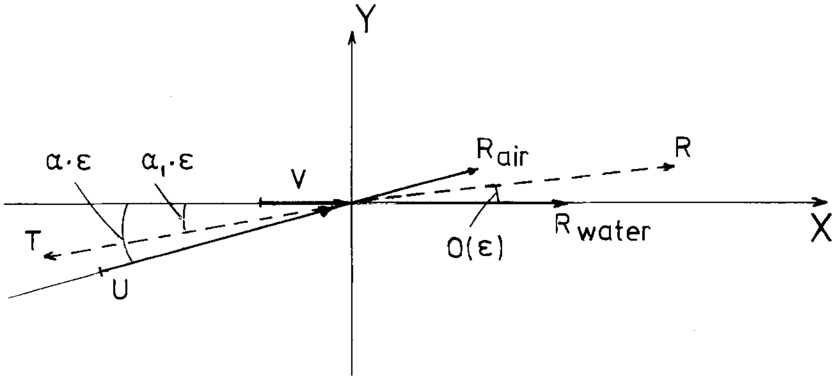

Figure 4.1. The directions of T and R.

where the additional righting moment $m^*(\beta)$ of the yacht has been written as

$$m^*(\beta) = \varepsilon\rho^\omega V^2 c^3 m(\beta), \tag{4.4}$$

where $m(\beta)$ is now a dimensionless function of the angle of heel β.

With the aid of (4.2) and (4.3), μ_1 and μ_2 can be expressed in v_1, v_2 and β. Substituting the results in formula (4.1), we have for the thrust:

$$T = -\tfrac{1}{2}\varepsilon^2\rho^\omega V^2 c^2 \cos^2\beta\Bigg[v_1^2 t_1 + v_1 v_2 t_2 + v_2^2 t_3 + v_1\left(t_4 \frac{m(\beta)}{\cos\beta} + 2\alpha I_{10}^\omega\right) + v_2 \frac{m(\beta)}{\cos\beta} t_5 + \left(\frac{m(\beta)}{\cos\beta}\right)^2 t_6\Bigg], \tag{4.5}$$

where

$$\begin{aligned} t_1 &= (I_{10}^\omega)^2\left\{PS^2L^2\frac{I_{21}^a}{D^a} + \frac{I_{21}^\omega}{D^\omega}\right\}, \quad P = \frac{\rho^\omega}{\rho^a}, \quad S = \frac{V}{U}, \quad L = \frac{c}{b-a},\\ t_2 &= I_{10}^\omega I_{11}^\omega\left\{PS^2L^3\frac{I_{11}^a + I_{20}^a}{D^a} - \frac{I_{11}^\omega + I_{20}^\omega}{D^\omega}\right\},\\ t_3 &= (I_{11}^\omega)^2\left\{PS^2L^4\frac{I_{10}^a}{D^a} + \frac{I_{10}^\omega}{D^\omega}\right\}, \quad t_4 = I_{10}^\omega PS^2L^3\frac{I_{11}^a + I_{20}^a}{D^a},\\ t_5 &= 2\frac{I_{10}^a I_{11}^\omega}{D^a}PS^2L^4, \quad t_6 = \frac{I_{10}^a}{D^a}PS^2L^4. \end{aligned} \tag{4.6}$$

From (4.5) we see that the angle α_1 does not appear in the optimum thrust formula. So in any direction the maximum thrust delivered by both the sails and the keel is the same, provided we are still in the $O(\varepsilon)$-region. This result could be expected. For in the equilibrium state with respect to the sideforces, the sails and the keel deliver the same sideforce, only in opposite direction, say F_s^*. Now from formula (3.19) it is clear that the part of the total thrust, which could depend on α_1, is $\alpha_1 F_s^* + (\alpha - \alpha_1)F_s^* = \alpha F_s^*$.

A question which might be important for constructional reasons is which part of the thrust is delivered by the sails and which part by the keel. As a consequence of the foregoing this question cannot be answered, unless we know the direction of the total thrust, and therefore the direction of the total resistance.

5. Optimization of T for given angle of heel

In order to get insight in the whole optimization problem, we consider first the case in which β is fixed. T can now be maximized as a function of the remaining free variables v_1 and v_2. Necessary conditions for the existence of an extreme of T are:

$$\frac{\partial T}{\partial v_1} = 0 \Rightarrow 2v_1 t_1 + v_2 t_2 + \left(t_4\frac{m(\beta)}{\cos\beta} + 2\alpha I_{10}^\omega\right) = 0, \tag{5.1}$$

$$\frac{\partial T}{\partial v_2} = 0 \Rightarrow v_1 t_2 + 2v_2 t_3 + \frac{m(\beta)}{\cos\beta} t_5 = 0. \tag{5.2}$$

These equations are satisfied for

$$v_1 = \frac{\dfrac{m(\beta)}{\cos\beta}(t_2 t_5 - 2t_3 t_4) - 4\alpha I_{10}^{\omega} t_3}{4t_1 t_3 - t_2^2},$$
$$v_2 = \frac{\dfrac{m(\beta)}{\cos\beta}(t_2 t_4 - 2t_1 t_5) + 2\alpha I_{10}^{\omega} t_2}{4t_1 t_3 - t_2^2}. \tag{5.3}$$

Sufficient for the existence of a maximum of T in this point is

$$\frac{\partial^2 T}{\partial v_1^2}\frac{\partial^2 T}{\partial v_2^2} - \left(\frac{\partial^2 T}{\partial v_1 \partial v_2}\right)^2 > 0 \;\; and \;\; \frac{\partial^2 T}{\partial v_1^2} < 0,$$

so we have to satisfy both the inequalities:

$$4t_1 t_3 - t_2^2 > 0 \;\; and \;\; t_1 > 0. \tag{5.4}$$

The inequalities (5.4) lead to relations between the solutions of the boundary value problems, as defined in (3.8). To prove these analytically seems very complicated, but anyway we can check them numerically.

On "physical" grounds it is clear that a maximum of T as a function of v_1 and v_2 should exist. For example, let $v_1^2 + v_2^2 \to \infty$. This means that the absolute values of the sideforces and/or heeling moments tend to infinity. This can only be achieved by increasing the circulation of the lifting lines, but this results in an increase of the induced resistance. So the thrust tends towards $-\infty$. Now the stationary point of T is given in (5.3), so there T should reach its maximum value.

The maximum thrust T, expressed as a function of β is:

$$T = -\tfrac{1}{2}\varepsilon^2 \rho^{\omega} V^2 c^2 [(m(\beta))^2 k_1 + \alpha m(\beta) k_2 \cos\beta + \alpha^2 k_3 \cos^2\beta], \tag{5.5}$$

where

$$k_1 = \{t_6(4t_1 t_3 - t_2^2) - t_1 t_5^2 + t_2 t_4 t_5 - t_3 t_4^2\}/(4t_1 t_3 - t_2^2),$$
$$k_2 = I_{10}^{\omega}(2t_2 t_5 - 4t_3 t_4)/(4t_1 t_3 - t_2^2), \tag{5.6}$$
$$k_3 = -4(I_{10}^{\omega})^2 t_3/(4t_1 t_3 - t_2^2).$$

If we put $\alpha = 0$ in (5.5), we find $T = -\frac{1}{2}\varepsilon^2\rho^{\omega}V^2c^2(m(\beta))^2 k_1$, and because a lifting line cannot produce a force in the opposite direction of the incoming flow, we may conclude that this T is the minimum induced resistance, and thus $k_1 > 0$. Furthermore from (5.4) we find $k_3 < 0$, hence

$$k_1 > 0, \quad k_3 < 0. \tag{5.7}$$

6. Approximate dependence of T on the righting moment

The remaining variable with respect to which we have to optimize the thrust is the heeling angle β. To be able to do this we have to know the righting moment m (4.4) as a function of β. For a given yacht in general there will not be a simple functional relation between m and β, so the optimum of T has to be found by numerical means. Here we will consider some simple righting moments, which may be first approximations for certain types of yachts, and for which the optimum β can be found approximately by analytical means.

For the optimization we need formula (5.5), which is a rather complicated one, because also the "factors" k_i ($i = 1, 2, 3$) in (5.6) are functions of β. However, in the region of interest for sailing, viz. $0 \leqq \beta \leqq 40°$, the k_i vary only slightly, because here, roughly speaking, the reflection by the watersurface is not yet sensed so strongly. In Fig. 6.1 this is illustrated for a special set of parameters S (4.6), L (4.6), α and gap a/l.

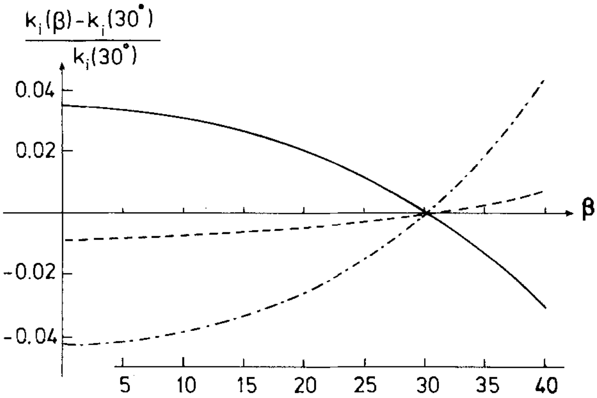

Figure 6.1. Variation of the k_i, proportional to their values for $\beta = 30°$; $S = \frac{1}{4}$, $L = \frac{1}{7}$, $\alpha = 20°$, gap 1%. ——: k_1 – – – –: k_2 –·–·–·–: k_3.

In this section we will assume the k_i to be constant. Of course, we have to check if the found optimum β's are in the region $0 \leqq \beta \leqq 40°$. In the calculations of the next section, the "exact" optimum is found by numerical means, and compared with the "analytical" optimum.

Consider first the situation, where no righting moment $m(\beta)$ is applied. From (5.5) we find the maximum thrust as a function of β:

$$T = -\tfrac{1}{2}\varepsilon^2 \rho^\omega V^2 c^2 \alpha^2 k_3 \cos^2 \beta, \tag{6.1}$$

and from (5.7) we see that this T is positive and has a maximum for $\beta = 0$.

As a first approximation for a narrow beam yacht, we may assume that its transverse metacenter (see e.g. [8], p. 70) remains at the same place as long as the heeling angle does not become too large. Then $m(\beta)$ is represented by

$$m(\beta) = M \sin \beta, \tag{6.2}$$

where M is a dimensionless constant. This constant is fixed by the displacement of the yacht and its transverse metacentric height (i.e. the distance between the center of buoyancy and the transverse metacenter).

Substituting (6.2) into (5.5) and putting $dT/d\beta = 0$ yields for the optimum angle of heel:

$$\operatorname{tg} 2\beta = \alpha M k_2/(\alpha^2 k_3 - M^2 k_1), \tag{6.3}$$

with the optimum thrust:

$$T = -\tfrac{1}{2}\varepsilon^2 \rho^\omega V^2 c^2 \cdot \tfrac{1}{2}\{M^2 k_1 + \alpha^2 k_3 - \sqrt{\alpha^2 M^2 k_2^2 + (\alpha^2 k_3 - M^2 k_1)^2}\}. \tag{6.4}$$

We observe, that this T is greater than (or at least equal to) the optimum T, when no righting moment is applied (6.1). So application of a righting moment (6.2) is favourable.

In the third case we consider the righting moment m as an independent variable (i.e. for every angle of heel β it may assume every possible value). This can be realised for example in a dinghy, where the crew can alter the righting moment by changing their position. For greater yachts it can be done by using an outrigger with an appropriate weight connected to it.

The conditions $\partial T/\partial m = \partial T/\partial \beta = 0$ yield for the optimum m and β:

$$2mk_1 + \alpha k_2 \cos\beta = 0, \tag{6.5}$$

$$-\alpha \sin\beta(mk_2 + 2\alpha k_3 \cos\beta) = 0. \tag{6.6}$$

Equation (6.6) is satisfied if one of its two factors is zero. First $\sin\beta = 0$, which together with (6.5) gives:

$$\beta = 0, \quad m = -\tfrac{1}{2}\alpha k_2/k_1. \tag{6.7}$$

In order that the second solution of (6.6), viz. $mk_2 + 2\alpha k_3 \cos\beta = 0$, be possible, the determinant of this equation and (6.5) should be zero. This determinant is $\det = \alpha(4k_1 k_3 - k_2)$, and from (5.7) we see that $\det \neq 0$. Hence here the only solution is the trivial one $\cos\beta = m = 0$, which corresponds to zero thrust for $\beta = \frac{1}{2}\pi$.

Sufficient for a maximum in (6.7) is $(\partial^2 T/\partial\beta^2)(\partial^2 T/\partial m^2) - (\partial^2 T/\partial m \partial\beta)^2 > 0$ together with $\partial^2 T/\partial m^2 < 0$, so we have to satisfy both the conditions

$$k_2^2 - 4k_1 k_3 > 0, \quad k_1 > 0, \tag{6.8}$$

which are true, because of (5.7).

The maximum thrust in this point (6.7) is

$$T = -\tfrac{1}{2}\varepsilon^2 \rho^\omega V^2 c^2 \alpha^2 (k_3 - \tfrac{1}{4}k_2^2/k_1), \tag{6.9}$$

and this is, again by (5.7), positive. It is, as it has to be, greater than the maximum thrust when no righting moment is applied (6.1).

Although the found optimum angles of heel β are approximations, they can be used as a starting point in the numerical calculations to find the exact ones. In the next section we will use formula (6.3) for this purpose.

7. A numerical example

We will give some numerical results, based on the exact linearized theory, for a yacht which may be considered as a typical half-tonner. Its parameters, which are important for our calculations are:

$$\begin{aligned}&\text{displacement} \equiv \Delta = 4000 \text{ kg},\\ &\text{transverse metacentric height} \equiv GM = 0.85 \text{ m},\\ &\text{draft} = c \text{ (Section 2)} = 1.7 \text{ m},\\ &\text{height of mast above water} = b \text{ (Section 2)} = 12 \text{ m}.\end{aligned} \tag{7.1}$$

For this type of yacht it is reasonable to give its righting moment (anyway for $0 \leqq \beta \leqq \frac{1}{4}\pi$) by formula (6.2), so $m^*(\beta)$ in (4.4) becomes

$$m^*(\beta) = GM \cdot \Delta \cdot \sin \beta = 3400 \sin \beta. \tag{7.2}$$

The velocity of the true wind $\bar{U}$ (Section 2) is taken to be 9 m/s (19 knots). We *choose* the speed of the yacht to be constant throughout the calculations, namely $V = \frac{1}{3}\bar{U}$.

From Fig. 2.2, it is easy to see that between the true and relative windspeed ($\bar{U}$ and U, respectively), and the true and relative angle of incidence ($\bar{\alpha}$ and α, respectively), we have the following relations:

$$U \approx \bar{U} + V = \tfrac{4}{3}\bar{U} = 12 \text{ m/s}, \quad \varepsilon\alpha \approx \bar{U}(\bar{U} + V)^{-1}\varepsilon\bar{\alpha} = \tfrac{3}{4}\varepsilon\bar{\alpha}. \tag{7.3}$$

The parameters S and L (4.6) now become:

$$S = \tfrac{1}{4}, \quad L = \frac{1.7}{12} \approx \tfrac{1}{7}. \tag{7.4}$$

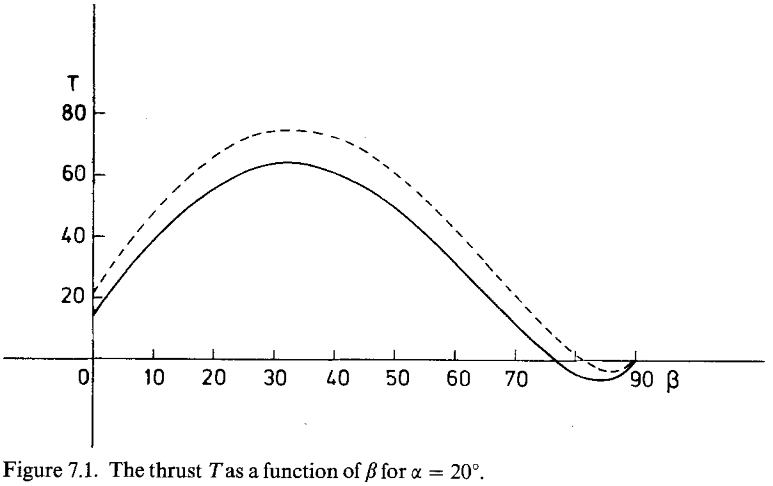

Figure 7.1. The thrust T as a function of β for $\alpha = 20°$.
——: gap 1% – – – –: no gap

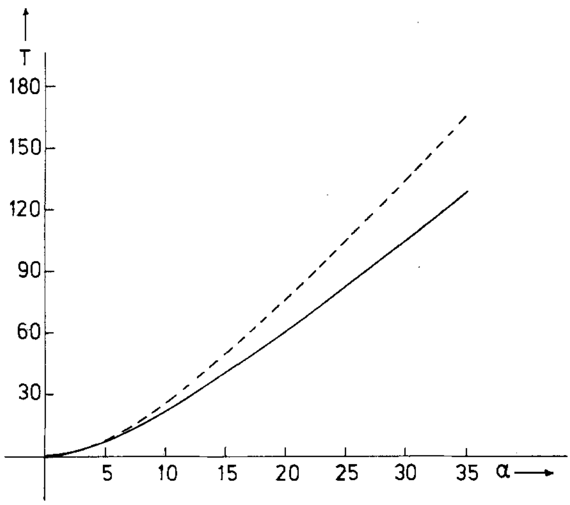

Figure 7.2. The optimum thrust.
——: gap 1% ----: no gap

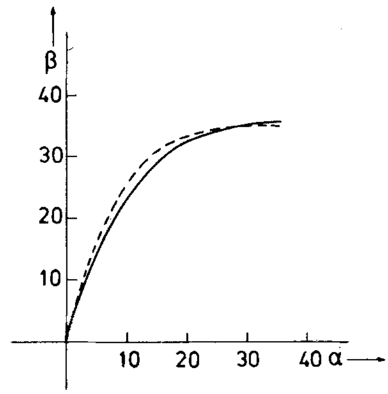

Figure 7.3. The optimum angle of heel.
——: gap 1% ----: no gap

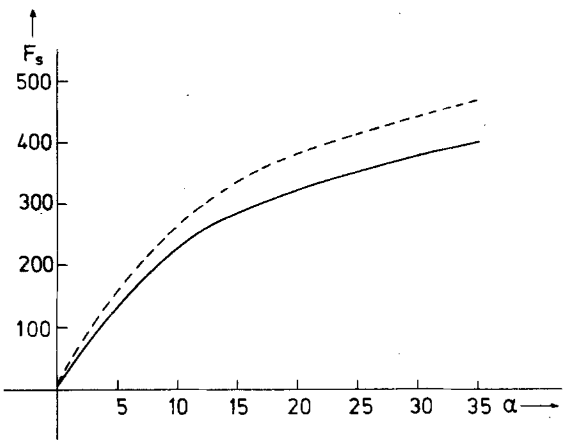

Figure 7.4. The sideforce, belonging to the optimum T.
——: gap 1% ----: no gap

For the gap we will consider two situations: no gap and a gap of 1% (= 12 cm between foresail and deck).

In this section, the forces will be given in kilogrammes, the angles in degrees.

Figure 7.1 shows the thrust T (5.5) (which has not yet been optimized with respect to β) as a function of the heeling angle β, with and without gap, for $\alpha = 20°$.

We remark that the formula (7.2) for the righting moment may need some corrections for large values of β. We observe that the difference in T for gap 1% and zero gap is about 15%. In [1] we saw that this difference was about 40%, in the case of no constraints, *so the application of constraints on sideforce and heeling moment has a strong levelling effect.* The occurrence of negative values of T for β in the neighbourhood of $\frac{1}{2}\pi$ is caused by the fact that a rather large heeling moment has to be produced in order to balance the righting moment which is almost equal to 3400 kgm in this region. Indeed, if no constraint was

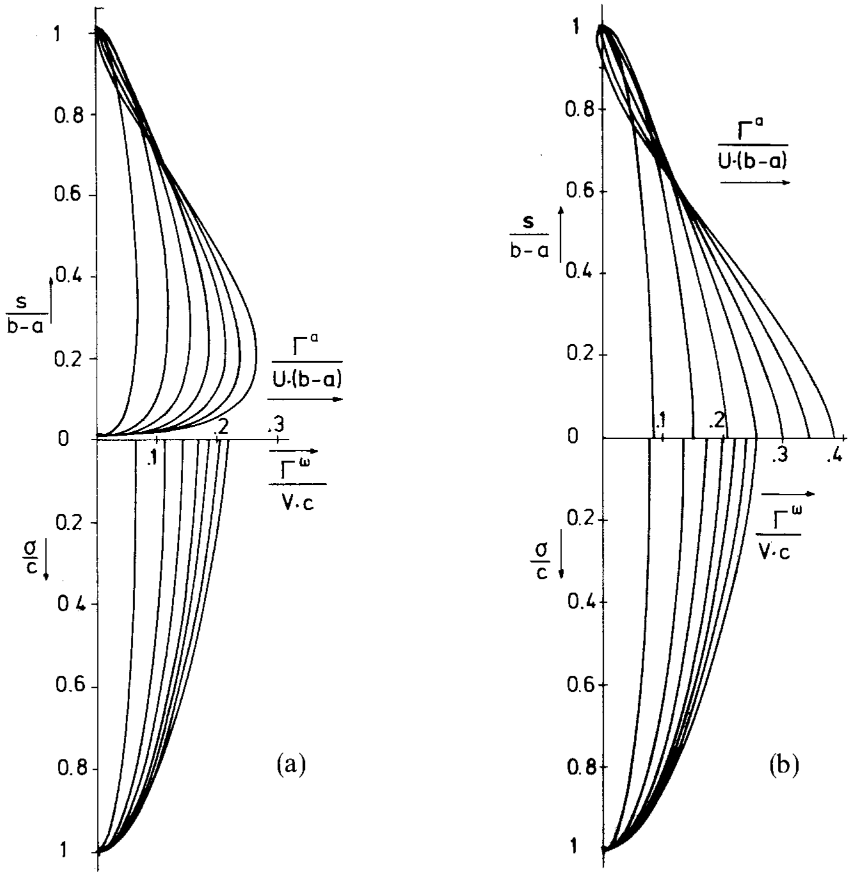

Figure 7.5a. The circulation distributions for $\alpha = 5, 10, \ldots, 35$; gap 1%.

Figure 7.5b. The circulation distributions for $\alpha = 5, 10, \ldots, 35$; no gap.

applied on the heeling moment in the region near $\beta = \frac{1}{2}\pi$, the lifting lines could always choose zero circulation which would give zero thrust. Then also the remaining constraint, viz. zero sideforce, would be satisfied.

Figures 7.2 to 7.4 give for $0 \leqq \alpha \leqq 35°$ the optimum thrust T, the angle of heel β for which this T is assumed, and the corresponding sideforce F_s (3.14). The values for T and β in Figs. 7.1 and 7.2 are the "exact" ones, however, the values, predicted by (6.4) and (6.3) showed an error of not more than 5%. Again we see a difference in the maximum thrust with and without gap of about 15%. Comparing Figs. 7.2 and 7.4, we see that the ratio between sideforce F_s and thrust T ranges from 22 for $\alpha = 5°$ to 3 for $\alpha = 35°$.

Remember (7.3) that in our example the real angle of incidence $\bar{\alpha}$ is about 1.3 times the relative one, α. So the region of interest is bounded below (from the practical viewpoint of sailing) by $\alpha \approx 20°$. We hope that this linearized theory will be valid for this value of α, and even for greater values, say until $\alpha = 30°$.

In figures 7.5a and 7.5b the optimum circulation distributions (3.16) are drawn for gap 1% and zero gap. The graphs in each set of diagrams with smallest maximum value belong to $\alpha = 5°$, the ones with next maximum value to $\alpha = 10°$, and so on.

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